On a new version of the Itô's formula for the stochastic heat equation

Alberto Lanconelli

Dipartimento di Matematica
Universita' degli Studi di Bari
Via E. Orabona, 4
70125 Bari - Italia
Email: lanconelli@dm.uniba.it

Abstract

We derive an Itô's-type formula for the one dimensional stochastic heat equation driven by a space-time white noise. The proof is based on elementary properties of the $\mathcal{S}$-transform and on the explicit representation of the solution process. We also discuss the relationship with other versions of this Itô's-type formula existing in literature.

Key words and phrases: stochastic heat equation, Itô's formula, S-transform, Wick product.
AMS 2000 classification: 60H15, 60H40.

1 Introduction

Consider the following stochastic partial differential equation (SPDE):

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2}\partial_{xx} u_t(x) + \dot{W}_{t,x} \\ u_t(0) = u_t(1) = 0, \quad u_0 = 0 \end{cases} \tag{1.1}$$

where $\partial_t := \frac{\partial}{\partial t}$, $\partial_{xx} := \frac{\partial^2}{\partial x^2}$, $\dot{W}_{t,x} := \frac{\partial^2 W_{t,x}}{\partial t \partial x}$ and $\{W_{t,x}, t \in [0,T], x \in [0,1]\}$ is a Brownian sheet. By solution to this equation we mean an adapted two parameter stochastic process $\{u_t(x), t \in [0,T], x \in [0,1]\}$ such that for $t \in [0,T]$ and $x \in [0,1]$,

$$u_t(0) = u_t(1) = 0, \quad u_0(x) = 0,$$

and such that for all $l \in C_0^2(]0,1[)$ the following equality

$$\langle u_t, l \rangle = \langle u_0, l \rangle + \frac{1}{2}\int_0^t \langle u_s, l'' \rangle ds + \int_0^t \langle l, dW_s \rangle,$$

holds for $t \in [0,T]$. Here $\langle , \rangle$ denotes the inner product in $\mathcal{L}^2([0,1])$ and

$$\int_0^t \langle l, dW_s \rangle := \int_0^t \int_0^1 l(x) dW_{s,x}.$$

It is well known (see e.g. [8]) that equation (1.1) has a unique solution $\{u_t(x), t \in [0,T], x \in [0,1]\}$ which is continuous in the variables (t,x) and that it can be represented as

$$u_t(x) = \int_0^t \int_0^1 g_{t-s}(x,y) dW_{s,y}$$

where $\{g_t(x,y), t \in [0,T], x,y \in [0,1]\}$ is the fundamental solution of the heat equation with homogenous Dirichlet boundary conditions, i.e.

$$\begin{cases} \partial_t g_t(x,y) = \frac{1}{2}\partial_{xx} g_t(x,y) \\ g_t(0,y) = g_t(1,y) = 0, \quad g_0(x,y) = \delta(x-y) \end{cases} \tag{1.2}$$

Since for fixed $x \in [0,1]$ the process $t \mapsto u_t(x)$ is not a semimartingale, we can not deal with it by means of the classical stochastic calculus. It is therefore natural to ask whether an Itô's-type formula can be found for this kind of process. Two recent papers, [2] and [9], are devoted to the investigation of this problem. In [2] the authors develop a Malliavin calculus for the solution process $u_t(x)$ in order to obtain an Itô's-type formula whose proof makes also use of projections on Wiener chaoses of different orders. In [9] the author applies the ordinary Itô's formula to a regularized version of the solution process $u_t(x)$; then he studies the limit when that regularized process converges to the real one. The main feature of this procedure is the appearance of a renormalization of the square of an infinite dimensional stochastic distribution.
The aim of the present paper is to propose an alternative approach (and a corresponding version of the Itô's-type formula) to the above mentioned problem which is somehow in between the techniques of the articles [2] and [9]; in fact we utilize notions of white noise analysis, an infinite dimensional stochastic distributions theory, and prove the resulting formula via scalar products with test functions, more precisely stochastic exponentials. The key point is the gaussianity of the solution process and its explicit representation as a stochastic convolution. Our formula is in the spirit of the Itô's-type formula for gaussian processes derived in the paper [7]. Moreover the idea of using the properties of the semigroup associated to the one dimensional Brownian motion (see the proof of Theorem 2.2) is analogous to the one used in [5].
The paper is structured as follows: in Section 2.1 we recall basic notions and facts from the white noise theory; then in Section 2.2 we state and prove our main result; finally in the concluding section we compare our formula with those already existing in the literature.

2 Main result

2.1 Preliminaries

In this section we fix the notation and recall some basic results from the white noise theory. For additional information about this topic we refer the reader to the books [3] and [4] and to the paper [1].
Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space and $\{W_{t,x}, t \in [0,T], x \in [0,1]\}$ a Brownian sheet defined on it. Assume that

$$\mathcal{F} = \sigma(W_{t,x}, t \in [0,T], x \in [0,1]),$$

so that each element $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ can be decomposed as a sum of multiple Itô integrals w.r.t. the Brownian sheet W, i.e.

$$X = \sum_{n \geq 0} I_n(h_n) \text{ (convergence in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})),$$

where for $n \geq 0$, h_n is a deterministic function belonging to $\mathcal{L}^2(([0,T] \times [0,1])^n)$ and $I_n(h_n)$ is the n-th order multiple Itô integral of h_n.
For example, the solution of the SPDE (1.1) has the decomposition:

$$u_t(x) = \int_0^t \int_0^1 g_{t-s}(x,y) dW_{s,y} = I_1(1_{[0,t]}(\cdot) g_{t-\cdot}(x, \cdot)).$$

If $A : D(A) \subset \mathcal{L}^2([0,T] \times [0,1]) \to \mathcal{L}^2([0,T] \times [0,1])$ is the unbounded operator

$$Ah(t,x) := A_t A_x h(t,x) := \sqrt{-\frac{\partial^2}{\partial t^2}} \sqrt{-\frac{\partial^2}{\partial x^2}} h(t,x)$$

with periodic boundary conditions, we define its *second quantization* as the operator,

$$\begin{array}{rcl} \Gamma(A) : D(\Gamma(A)) \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) & \to & \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \\ \sum_{n \geq 0} I_n(h_n) & \mapsto & \Gamma(A)(\sum_{n \geq 0} I_n(h_n)) := \sum_{n \geq 0} I_n(A^{\otimes n} h_n). \end{array}$$

For $p \geq 1$ the space

$$(S)_p := \{X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \text{ s.t. } E[|\Gamma(A^p)X|^2] < +\infty\}$$

is a subset of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ and if $q > p$ then $(S)_q \subset (S)_p$. The *Hida's test functions space* is defined as

$$(S) := \bigcap_{p \geq 1} (S)_p.$$

Its dual w.r.t. the inner product of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is called *Hida's distributions space* and denoted by $(S)^*$. It can be shown that

$$(S)^* = \bigcup_{p \geq 1} (S)_{-p};$$

moreover by construction

$$(S) \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \subset (S)^*.$$

For $f \in C_0^\infty(]0,T[\times]0,1[)$, the random variable

$$\mathcal{E}_T(f) := \exp\{\int_0^T \int_0^1 f(s,y) dW_{s,y} - \frac{1}{2} \int_0^T \int_0^1 f^2(s,y) dy ds\},$$

belongs to (S); therefore if $\langle\langle , \rangle\rangle$ denotes the dual pairing between $(S)^*$ and (S) then the application

$$X \in (S)^* \mapsto \mathcal{S}(X)(f) := \langle\langle X, \mathcal{E}_T(f) \rangle\rangle$$

is well defined and it is called $\mathcal{S}$-*transform*. The quantity $\mathcal{S}(X)(f), f \in C_0^\infty(]0,T[\times]0,1[)$ identifies uniquely the Hida's distribution X. In particular, given $X, Y \in (S)^*$, we denote by $X \diamond Y$ the unique element of $(S)^*$ such that

$$\mathcal{S}(X \diamond Y)(f) = \mathcal{S}(X)(f)\mathcal{S}(Y)(f), \text{ for all } f \in C_0^\infty(]0,T[\times]0,1[);$$

the stochastic distribution $X \diamond Y$ is named *Wick product* of X and Y.
Notice also that if $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ then

$$\mathcal{S}(X)(f) = \langle\langle X, \mathcal{E}_T(f)\rangle\rangle = E[X\mathcal{E}_T(f)],$$

and if ξ is an Itô integrable stochastic process then

$$\mathcal{S}(\int_0^T \int_0^1 \xi_{t,x} dW_{t,x})(f) = E[\int_0^T \int_0^1 \xi_{t,x} dW_{t,x}\mathcal{E}_T(f)] = \int_0^T \int_0^1 E[\xi_{t,x}\mathcal{E}_T(f)]dxdt.$$

2.2 Itô's-type formula

Before the main theorem of this paper, we state and prove the following auxiliary result.

Lemma 2.1 *Let $u_t(x)$ be the solution of the SPDE (1.1); then $\partial_{xx}u_t(x) \in (S)^*$.*

PROOF. It is known that the fundamental solution $g_t(x,y)$ of the heat equation with homogenous Dirichlet boundary conditions (1.2) can be represented as

$$g_t(x,y) = \sum_{n\geq 1} e^{-\lambda_n t} e_n(x) e_n(y),$$

where $e_n(x) = \sqrt{2}sin(\pi n x)$ and $\lambda_n = \pi^2 n^2$. This gives

$$\begin{aligned}
\partial_{xx} g_t(x,y) &= \sum_{n\geq 1} e^{-\lambda_n t}(-\lambda_n) e_n(x) e_n(y) \\
&= \sum_{n\geq 1} e^{-\lambda_n t} e_n(x) \partial_{yy} e_n(y) \\
&= \sum_{n\geq 1} e^{-\lambda_n t} e_n(x)(-A_y^2 e_n(y)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[|\Gamma(A^{-2})\partial_{xx}u_t(x)|^2] &= \int_0^t \int_0^1 |A_s^{-2} A_y^{-2} \partial_{xx} g_{t-s}(x,y)|^2 dyds \\
&= \int_0^t \int_0^1 |A_s^{-2} g_{t-s}(x,y)|^2 dyds \\
&\leq \int_0^t \int_0^1 |g_{t-s}(x,y)|^2 dyds < +\infty,
\end{aligned}$$

proving that $\partial_{xx}u_t(x) \in (S)_{-2} \subset (S)^*$. □

Theorem 2.2 *For any $\varphi \in C_b^2(\mathbb{R})$ and $l \in C_0^2(]0,1[)$ one has*

$$\begin{aligned}\langle \varphi(u_t), l\rangle &= \langle \varphi(u_0), l\rangle + \int_0^t \langle \varphi'(u_s)l, dW_s\rangle + \frac{1}{2}\langle \int_0^t \varphi'(u_s) \diamond \partial_{xx} u_s ds, l\rangle \\ &+ \frac{1}{2}\langle \int_0^t \varphi''(u_s) d\sigma^2(s;\cdot), l\rangle, \end{aligned} \tag{2.1}$$

where

- $\sigma^2(s,x) = E[|u_s(x)|^2] = \int_0^s \int_0^1 g_{s-v}^2(x,y)dydv;$
- $\int_0^t \varphi''(u_s(x))d\sigma^2(s;x)$ *is a Stiltjes integral in the variable* s*;*
- $\varphi'(u_s) \diamond \partial_{xx} u_s$ *is the Wick product between* $\varphi'(u_s)$ *and* $\partial_{xx} u_s$*.*

Proof. We aim at proving that

$$E[\langle \varphi(u_t), l\rangle \mathcal{E}_T(f)] = E[\mathcal{R}\mathcal{E}_T(f)],$$

for all $f \in C_0^\infty(]0,T[\times]0,1[)$ where $\mathcal{R}$ denotes the r.h.s. of equation (2.1). This fact together with the property

$$span\{\mathcal{E}_T(f), f \in C_0^\infty(]0,T[\times]0,1[)\} \text{ is dense in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$$

will imply that

$$\langle \varphi(u_t), l\rangle = \mathcal{R} \quad \mathcal{P}-\text{almost surely},$$

which is the statement of the theorem.
Let us fix an arbitrary $f \in C_0^\infty(]0,T[\times]0,1[)$; a simple application of the Girsanov's theorem yields:

$$\begin{aligned} E[\varphi(u_t(x))\mathcal{E}_T(f)] &= E\Big[\varphi\Big(\int_0^t \int_0^1 g_{t-s}(x,y)dW_{s,y}\Big)\mathcal{E}_T(f)\Big] \\ &= E\Big[\varphi\Big(\int_0^t \int_0^1 g_{t-s}(x,y)dW_{s,y} + \int_0^t \int_0^1 g_{t-s}(x,y)f(s,y)dyds\Big)\Big]. \end{aligned}$$

Now observe that

$$\int_0^t \int_0^1 g_{t-s}(x,y)dW_{s,y} + \int_0^t \int_0^1 g_{t-s}(x,y)f(s,y)dyds,$$

is a Gaussian random variable with mean given by

$$m(t,x) := \int_0^t \int_0^1 g_{t-s}(x,y)f(s,y)dyds,$$

and variance equal to

$$\sigma^2(t,x) := \int_0^t \int_0^1 g_{t-s}^2(x,y)dyds.$$

Therefore we can write

$$\begin{aligned}
E[\varphi(u_t(x))\mathcal{E}_T(f)] &= E\Big[\varphi\Big(\int_0^t\int_0^1 g_{t-s}(x,y)dW_{s,y}+\int_0^t\int_0^1 g_{t-s}(x,y)f(s,y)dyds\Big)\Big] \\
&= (P_{\sigma^2(t,x)}\varphi)(m(t,x)), \qquad (2.2)
\end{aligned}$$

where $\{P_t\}_{t\geq 0}$ denotes the one dimensional heat semigroup,

$$(P_t\varphi)(x):=\int_{\mathbb{R}}\varphi(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}dy.$$

Identity (2.2) turn out to be very useful; in fact we can apply the ordinary chain rule and get

$$\begin{aligned}
E[\varphi(u_t(x))\mathcal{E}_T(f)] &= (P_{\sigma^2(t,x)}\varphi)(m(t,x)) \\
&= \varphi(u_0(x))+\int_0^t\frac{d}{dv}(P_{\sigma^2(v,x)}\varphi)(m(v,x))dv \\
&= \varphi(u_0(x))+\int_0^t(P_{\sigma^2(v,x)}\varphi')(m(v,x))\frac{dm(v,x)}{dv}dv \\
&+ \int_0^t\frac{1}{2}(P_{\sigma^2(v,x)}\varphi'')(m(v,x))d\sigma^2(v;x) \\
&=: \varphi(u_0(x))+\mathcal{I}(t,x)+\mathcal{II}(t,x).
\end{aligned}$$

Recalling that

$$m(v,x)=\int_0^v\int_0^1 g_{v-s}(x,y)f(s,y)dyds,$$

we have

$$\begin{aligned}
\frac{dm(v,x)}{dv} &= f(v,x)+\int_0^v\int_0^1\frac{1}{2}\partial_{xx}g_{v-s}(x,y)f(s,y)dyds \\
&= f(v,x)+\frac{1}{2}\partial_{xx}\Big(\int_0^v\int_0^1 g_{v-s}(x,y)f(s,y)dyds\Big),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{I}(t,x) &= \int_0^t(P_{\sigma^2(v,x)}\varphi')(m(v,x))\Big(f(v,x)+\frac{1}{2}\partial_{xx}\Big(\int_0^v\int_0^1 g_{v-s}(x,y)f(s,y)dyds\Big)\Big)dv \\
&= \int_0^t(P_{\sigma^2(v,x)}\varphi')(m(v,x))f(v,x)dv \\
&+ \int_0^t(P_{\sigma^2(v,x)}\varphi')(m(v,x))\frac{1}{2}\partial_{xx}\Big(\int_0^v\int_0^1 g_{v-s}(x,y)f(s,y)dyds\Big)dv \\
&= \int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]f(v,x)dv+\int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]\frac{1}{2}\partial_{xx}E[u_v(x)\mathcal{E}_T(f)]dv,
\end{aligned}$$

where the last equality is due to identities (2.2). Moreover since from Lemma 2.1 $\partial_{xx}u_v(x)\in(S)^*$ we have

$$\partial_{xx}E[u_v(x)\mathcal{E}_T(f)]=\partial_{xx}\mathcal{S}(u_v(x))(f)=\mathcal{S}(\partial_{xx}u_v(x))(f),$$

where $\mathcal{S}(\partial_{xx}u_v(x))(f)$ denotes the $\mathcal{S}$-transform of $\partial_{xx}u_v(x)$.
If now $l \in C_0^2(]0,1[)$ is a test function in the space variable x, by the properties of the Wick product we conclude that

$$\begin{aligned}
\int_0^1 \mathcal{I}(t,x)l(x)dx &= \int_0^1\int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]l(x)f(v,x)dvdx \\
&+ \int_0^1\int_0^t E[\varphi'(u_v)\mathcal{E}_T(f)]\frac{1}{2}\mathcal{S}(\partial_{xx}u_v(x))(f)l(x)dvdx \\
&= \int_0^1\int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]l(x)f(v,x)dvdx \\
&+ \int_0^1\int_0^t \mathcal{S}(\varphi'(u_v(x))\diamond\frac{1}{2}\partial_{xx}u_v(x))(f)l(x)dvdx \\
&= E\Big[\Big(\int_0^t\int_0^1 \varphi'(u_v(x))l(x)dW_{v,x} \\
&+ \int_0^1\Big(\int_0^t \varphi'(u_v(x))\diamond\frac{1}{2}\partial_{xx}u_v(x)dv\Big)l(x)dx\Big)\mathcal{E}_T(f)\Big].
\end{aligned}$$

Looking again at identities (2.2) we also discover that

$$\mathcal{II}(t,x) = \int_0^t \frac{1}{2}E[\varphi''(u_s(x))\mathcal{E}_T(f)]d\sigma^2(s;x);$$

combining the expressions for $\mathcal{I}$ and $\mathcal{II}$ we can now conclude that

$$\begin{aligned}
\langle E[\varphi(u_t)\mathcal{E}_T(f)],l\rangle &= \langle\varphi(u_0),l\rangle \\
&+ E\Big[\Big(\int_0^t\int_0^1 \varphi'(u_v(x))l(x)dW_{v,x} + \langle\int_0^t \varphi'(u_v)\diamond\frac{1}{2}\partial_{xx}u_v dv, l\rangle\Big)\mathcal{E}_T(f)\Big] \\
&+ \langle\int_0^t \frac{1}{2}E[\varphi''(u_s)\mathcal{E}_T(f)]d\sigma^2(v;\cdot),l\rangle \\
&= \langle\varphi(u_0),l\rangle \\
&+ E\Big[\Big(\int_0^t \langle\varphi'(u_v)l,dW_v\rangle + \langle\int_0^t \varphi'(u_v)\diamond\frac{1}{2}\partial_{xx}u_v dv, l\rangle\Big)\mathcal{E}_T(f)\Big] \\
&+ E\Big[\Big(\langle\frac{1}{2}\int_0^t \varphi''(u_s)d\sigma^2(v;\cdot),l\rangle\Big)\mathcal{E}_T(f)\Big].
\end{aligned}$$

This completes the proof. □

2.3 Comparisons

Zambotti's formula : In [9] the author obtains the following Itô's formula:

$$\begin{aligned}
\langle\varphi(u_t),l\rangle &= \langle\varphi(u_0),l\rangle + \frac{1}{2}\int_0^t \langle l'',\varphi(u_s)\rangle ds + \int_0^t \langle\varphi'(u_s)l,dW_s\rangle \\
&-\frac{1}{2}\int_0^t \langle l, :\left|\partial_x u_s\right|^2 : \varphi''(u_s)\rangle ds, \qquad (2.3)
\end{aligned}$$

where the last term is defined as the limit of renormalized diverging quantities. The procedure to derive this formula is to approximate the solution of the SPDE via the smoother process

$$u_t^\epsilon(x) := \int_0^t \int_0^1 g_{t-s+\epsilon}(x,y)dW_{s,y},$$

and then to pass to the limit for $\epsilon \to 0$. We are now going to show how to manipulate formula (2.1) to make it look like (2.3).
Following the same line of reasoning explained in the proof of Theorem 2.2 with $u_t^\epsilon(x)$ instead of $u_t(x)$, one can easily see that everything can be carried in a similar manner; the only difference consists in the fact that in this case the function

$$\sigma_\epsilon^2(t,x) := E[|u_t^\epsilon(x)|^2] = \int_0^t \int_0^1 g_{t-s+\epsilon}^2(x,y)dyds,$$

is differentiable w.r.t. t; therefore the last term of formula (2.1) can be rewritten as

$$\begin{aligned}
\frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))d\sigma_\epsilon^2(s;x) \quad &= \quad \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\frac{d\sigma_\epsilon^2(s,x)}{ds}ds \\
= \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\Big(\int_0^1 g_\epsilon^2(x,y)dy \quad &+ \quad \int_0^s\int_0^1 \partial_s(g_{s-v+\epsilon}^2(x,y))dydv\Big)ds \\
= \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\Big(\int_0^1 g_\epsilon^2(x,y)dy \quad &+ \quad \int_0^s\int_0^1 g_{s-v+\epsilon}(x,y)\partial_{xx}g_{s-v+\epsilon}(x,y)dydv\Big)ds \\
= \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\int_0^1 g_\epsilon^2(x,y)dyds \quad &+ \quad \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\int_0^s\int_0^1 g_{s-v+\epsilon}(x,y)\partial_{xx}g_{s-v+\epsilon}(x,y)dydvds \\
= \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\int_0^1 g_\epsilon^2(x,y)dyds \quad &+ \quad \frac{1}{2}\int_0^t\int_0^s\int_0^1 D_{v,y}\varphi'(u_s^\epsilon(x))D_{v,y}\partial_{xx}u_s^\epsilon(x)dydvds \\
= \frac{1}{2}\int_0^t \varphi''(u_s^\epsilon(x))\int_0^1 g_\epsilon^2(x,y)dyds \quad &+ \quad \frac{1}{2}\int_0^t \varphi'(u_s^\epsilon(x))\partial_{xx}u_s^\epsilon(x) - \varphi'(u_s^\epsilon(x))\diamond\partial_{xx}u_s^\epsilon(x)ds.
\end{aligned}$$

Here we used a property of the Wick product which shows its interplay with the Hida-Malliavin derivative, namely

$$X \diamond \int_0^T\int_0^1 h(s,y)dW_{s,y} = X\int_0^T\int_0^1 h(s,y)dW_{s,y} - \int_0^T\int_0^1 D_{s,y}Xh(s,y)dyds.$$

See [4] and [6] for details. Therefore equation (2.1) becomes

$$\begin{aligned}
\langle\varphi(u_t^\epsilon),l\rangle \quad &= \quad \langle\varphi(u_0^\epsilon),l\rangle + \int_0^t \langle\varphi'(u_s^\epsilon)l,dW_s\rangle + \frac{1}{2}\langle\int_0^t \varphi'(u_s^\epsilon)\diamond\partial_{xx}u_s^\epsilon ds,l\rangle \\
&\quad +\frac{1}{2}\langle\int_0^t \varphi''(u_s^\epsilon)\int_0^1 g_\epsilon^2(\cdot,y)dyds,l\rangle \\
&\quad +\frac{1}{2}\langle\int_0^t \varphi'(u_s^\epsilon)\partial_{xx}u_s^\epsilon - \varphi'(u_s^\epsilon)\diamond\partial_{xx}u_s^\epsilon ds,l\rangle \\
&= \quad \langle\varphi(u_0^\epsilon),l\rangle + \int_0^t \langle\varphi'(u_s^\epsilon)l,dW_s\rangle \\
&\quad +\frac{1}{2}\langle\int_0^t \varphi''(u_s^\epsilon)\int_0^1 g_\epsilon^2(\cdot,y)dyds,l\rangle + \frac{1}{2}\langle\int_0^t \varphi'(u_s^\epsilon)\partial_{xx}u_s^\epsilon ds,l\rangle.
\end{aligned}$$

Moreover

$$\varphi'(u_s^\epsilon(x))\partial_{xx}u_s^\epsilon(x) = \partial_{xx}\varphi(u_s^\epsilon(x)) - \varphi''(u_s^\epsilon(x))(\partial_x u_s^\epsilon(x))^2;$$

a substitution in the previous equation gives

$$\begin{aligned}
\langle\varphi(u_t^\epsilon), l\rangle &= \langle\varphi(u_0^\epsilon), l\rangle + \int_0^t \langle\varphi'(u_s^\epsilon)l, dW_s\rangle + \frac{1}{2}\langle\int_0^t \varphi''(u_s^\epsilon)(\int_0^1 g_\epsilon^2(\cdot, y)dy)ds, l\rangle \\
&+\frac{1}{2}\langle\int_0^t \partial_{xx}\varphi(u_s^\epsilon) - \varphi''(u_s^\epsilon)(\partial_x u_s^\epsilon)^2 ds, l\rangle \\
&= \langle\varphi(u_0^\epsilon), l\rangle + \int_0^t \langle\varphi'(u_s^\epsilon)l, dW_s\rangle - \frac{1}{2}\langle\int_0^t \varphi''(u_s^\epsilon)((\partial_x u_s^\epsilon)^2 - \int_0^1 g_\epsilon^2(\cdot, y)dy)ds, l\rangle \\
&+\frac{1}{2}\langle\int_0^t \varphi(u_s^\epsilon), l''\rangle.
\end{aligned}$$

Since this identity coincides with the expression derived in [9] before taking the limit for $\epsilon \to 0$, the equivalence of the two formulas is proved.

Gradinaru-Nourdin-Tindel's formula : In [2] the authors present the following Itô's formula:

$$\varphi(u_t) = \varphi(u_0) + \int_0^t \langle\varphi'(u_s), \delta u_s\rangle + \frac{1}{2}\int_0^t Tr(e^{2s\Delta}\varphi''(u_s))ds. \tag{2.4}$$

The term $\int_0^t \langle\varphi'(u_s), \delta u_s\rangle$ denotes a Skorohod's type integral w.r.t. the solution process $u_t(x)$. If we write formally

$$\delta u_s(x) = \frac{1}{2}\partial_{xx}u_s(x)ds + \delta W_{s,x} \text{ and } \int_0^t \langle\varphi'(u_s), \delta u_s\rangle = \int_0^t\int_0^1 \varphi'(u_s) \diamond \frac{du_s(x)}{ds}dxds,$$

we get

$$\begin{aligned}
\int_0^t \langle\varphi'(u_s), \delta u_s\rangle &= \int_0^t\int_0^1 \varphi'(u_s) \diamond \frac{du_s(x)}{ds}dxds \\
&= \int_0^t\int_0^1 \varphi'(u_s) \diamond \Big(\frac{1}{2}\partial_{xx}u_s(x) + \dot{W}_{s,x}\Big)dxds \\
&= \int_0^t\int_0^1 \varphi'(u_s) \diamond \frac{1}{2}\partial_{xx}u_s(x)dxds + \int_0^t\int_0^1 \varphi'(u_s)dW_{s,x}.
\end{aligned}$$

This procedure, far from being rigorous, suggests some common feature between (2.1) and (2.4). However the identification of the "Itô's terms" of the two formulas, namely

$$\int_0^t \varphi''(u_s(\cdot))d\sigma^2(s;\cdot) \text{ and } \int_0^t Tr(e^{2s\Delta}\varphi''(u_s))ds,$$

doesn't seem to be straightforward.

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